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A GENERALIZED SCHWARZ INEQUALITY AND ALGEBRAIC INVARIANTS FOR OPERATOR ALGEBRAS

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1. Introduction

In [2][2] we classified the isometric mappings of one C^*-algebra (uniformly closed, self-adjoint operator algebra) onto another. It was remarked in that paper that the results obtained were a non-commutative extension of results of Banach [1] and Stone [7]. While this was true in spirit, we were well aware that it was not accurate to the letter. Banach and Stone deal with the algebra of *real* continuous functions on a compact-Hausdorff space, and our results concerning C^*-algebras are actually the non-commutative analogue of results concerning the *complex* function algebra. The strict non-commutative analogue of the real function algebra is the Jordan algebra of self-adjoint elements in a C^*-algebra (Jordan C^*-algebra). The complex and real theorems follow very easily from one another in the commutative case, so that one might justifiably consider the C^*-algebra theorem an extension of both of the function algebra theorems. Despite such trifling considerations, two questions still remain: what are the isometries of one C^*-algebra onto another, and what are the isometries of one Jordan C^*-algebra onto another? At the time [2] was written, the C^*-algebra seemed the more natural object to consider. In view of the results obtained, answering the Jordan C^*-algebra questions appeared to be an unnecessary decoration to the theory. We felt that the Jordan C^*-algebra results could be obtained from the C^*-algebra results in the same way that the real function algebra theorem follows from the complex function algebra theorem (viz., by showing that the complexified linear map is everywhere isometric). Subsequent investigations have changed our attitude in this matter. An important application of these considerations requires a Jordan C^*-algebra theorem for one thing, and our attempts to derive this theorem directly from the C^*-algebra theorem failed for another.

The result in question is contained in Theorem 2 of §2 and states (in normalized form) that an isometry between two Jordan C^*-algebras which carries the identity into the identity is a C^* (Jordan) -isomorphism. This theorem was eventually proved with the aid of a Generalized Schwarz Inequality (cf. Theorem 1 of §2). In effect, an alternative ending has been given to the proof of [Theorem 7; 2]. This ending is by no means simpler or shorter than the one given in [2] (though it is, perhaps, less contrived), but it is flexible enough to allow us to draw the desired Jordan C^*-algebra conclusion.

The critical application of these results is contained in Corollary 3. A discussion accompanies Corollaries 3 and 4, but a few additional remarks are in order.

[1] This paper was written while the author was a National Research Fellow.

[2] Numbers in brackets refer to the bibliography at the end of this paper.

Experience with C^*-algebras shows them to be quite untamed as a general class of algebras. It may well be that the algebraic invariants we attach to C^*-algebras carry the question of the algebraic nature of C^*-algebras as far as it can go in general terms. These invariants have the merit of being a very natural extension of the invariants one has in the commutative case. The task left in investigating a particular C^*-algebra or a particular class of C^*-algebras is the computation of the pure states.

Section 2 contains, in addition to the assertions indicated, several related results. Section 3, the concluding section, contains some examples and remarks which settle many of the questions brought up by the results of §2.

2. The principal results

The following theorem is the main step toward our final results.

THEOREM 1 (The Generalized Schwarz Inequality). *Let $\mathfrak{A}$ be a C^*-algebra, and let ϕ be a linear order preserving map of $\mathfrak{A}$ into the algebra of all bounded operators on some Hilbert space such that $\|\phi\| \leqq 1$. Then $\phi(A^2)' \geqq \phi(A)^2$ for each self-adjoint operator A in $\mathfrak{A}$.*

PROOF. Since the theorem deals only with the real algebra generated by a single self-adjoint operator A and I at one time, we may restrict our attention to such algebras (as domain algebras for ϕ). Such algebras are (algebraically) isomorphic to the set of all continuous (real) functions on the spectrum of the operator A, and thus we can consider maps ϕ of the described type from $C(X)$ to bounded operators (where X is a compact subset of the reals). We denote by the same symbol, the operator and its representing function. Let $E_1, \cdots, E_n$ be characteristic functions of open subsets and closed subsets of X such that $E_iE_j = 0$ and such that $\sum\alpha_iE_i$ is close to A uniformly. We shall extend ϕ, preserving its order and norm properties, to the space generated by $C(X)$ and the functions $E_1, \cdots, E_n$. Suppose, for the moment, that this is done, and suppose, also, that

$$\phi[(\textstyle\sum\alpha_iE_i)^2] \geqq [\phi(\textstyle\sum\alpha_iE_i)]^2. \tag{1}$$

By uniform continuity of ϕ, we have that $\phi[(\sum\alpha_iE_i)^2]$ is close to $\phi(A^2)$ and $[\phi(\sum\alpha_iE_i)]^2$ is close to $\phi(A)^2$. The arbitrariness of the degree of approximation implies that $\phi(A^2) \geqq \phi(A)^2$. It remains to prove that ϕ can be so extended and that (1) holds.

We prove, first, the inequality (1). Letting $\phi(E_i) = A_i$, (1) becomes $\sum\alpha_i^2A_i \geqq (\sum\alpha_iA_i)^2$. To prove this last inequality, we must show that

$$((\textstyle\sum\alpha_i^2A_i)x, x) \geqq ((\textstyle\sum\alpha_iA_i)^2x, x) = \|\textstyle\sum\alpha_iA_ix\|^2 \tag{2}$$

for all x in the Hilbert space. We note that since $E_i \geqq 0$ and $\sum E_i \leqq 1$, we have $A_i \geqq 0$ and $\sum A_i \leqq I$. We shall prove a more general inequality than (2), viz.,

$$\textstyle\sum(A_iy_i, y_i) \geqq \|\textstyle\sum A_iy_i\|^2 \tag{3}$$

when the positive operators A_i satisfy the condition $\sum A_i \leqq I$. The inequality (3) yields (2) when we set $\alpha_ix = y_i$. To prove (3) we consider the direct sum of n copies of the original Hilbert space and introduce the positive semi-definite inner product

$$[(z_1, \cdots, z_n), (x_1, \cdots, x_n)] = \sum (A_i z_i, x_i).$$

The Schwarz Inequality holds for this inner product, and we apply it to the vectors $(z, \cdots, z)$, $(y_1, \cdots, y_n)$ where $z = \sum A_i y_i$. This gives

$$\begin{aligned} \| z \|^2 &= \textstyle\sum_{i,j} (A_j A_i y_i, y_j) = [(z, \cdots, z), (y_1, \cdots, y_n)] \\ &\leqq [(z, \cdots, z), (z, \cdots, z)]^{\frac{1}{2}} [(y_1, \cdots, y_n), (y_1, \cdots, y_n)]^{\frac{1}{2}} \\ &= (\textstyle\sum A_i z, z)^{\frac{1}{2}} (\textstyle\sum (A_i y_i, y_i))^{\frac{1}{2}} \leqq (z, z)^{\frac{1}{2}} (\textstyle\sum (A_i y_i, y_i))^{\frac{1}{2}} \\ &= \| z \| (\textstyle\sum (A_i y_i, y_i))^{\frac{1}{2}} \end{aligned}$$

from which (3) results.

We show now that ϕ can be extended. For this purpose, we pattern our argument after that which establishes the Riesz representation theorem for linear functionals on $C(X)$. In fact if E_1 is the characteristic function of an open set, we choose a monotone increasing sequence (A_n) of continuous functions which approach E_1 pointwise (if E_1 is the characteristic function of a closed set, we deal with $1 - E_1$). Then $(\phi(A_n))$ is a monotone increasing sequence of operators bounded above by $\phi(I)$ which, according to [8], has a strong limit $\phi(E_1)$. As defined, $\phi(E_1)$ is unique. In fact, let (B_n) be another sequence of functions with the same properties as (A_n). Denote the strong limit of $(\phi(B_n))$ by $\phi(E_1)'$, and let x be a vector in the underlying Hilbert space. We show that $\phi(E_1)' = \phi(E_1)$ by showing that $(\phi(E_1)'x, x) = (\phi(E_1)x, x)$. Indeed $(\phi(E_1)'x, x) = \lim_n(\phi(B_n)x, x)$ and $(\phi(E_1)x, x) = \lim_n(\phi(A_n)x, x)$. Now, by the Riesz representation theorem, the positive functional $(\phi(\cdot)x, x)$ on $C(X)$ defines a measure μ on X, and, by the monotone convergence theorem, $\lim_n(\phi(A_n)x, x) = \mu(E_1) = \lim_n(\phi(B_n)x, x)$, from which the uniqueness of $\phi(E_1)$ follows (this proves that if E_1 happens to be in $C(X)$, the old and new definitions of $\phi(E_1)$ agree). We extend ϕ by linearity to the space generated by $E_1, \cdots, E_n$ and $C(X)$, i.e., to all functions of the form $\alpha_1 E_1 + \cdots + \alpha_n E_n + B$ with B in $C(X)$. That ϕ so extended is uniquely defined follows from the fact that if $\alpha_1 E_1 + \cdots + \alpha_n E_n + B$ represents 0 then $\alpha_1 A_m^{(1)} + \cdots + \alpha_n A_m^{(n)} + B$ is dominated by some constant (for all m) and tends pointwise to 0, where $A_m^{(i)}$ (or $1 - A_m^{(i)}$) was used to define $\phi(E_i)$. Thus, by the dominated convergence theorem,

$$\begin{aligned} &(\phi(\alpha_1 A_m^{(1)} + \cdots + \alpha_n A_m^{(n)} + B)x, x) \\ &\qquad = ([\alpha_1 \phi(A_m^{(1)}) + \cdots + \alpha_n \phi(A_m^{(n)}) + \phi(B)]x, x) \end{aligned}$$

tends to 0. This limit is, however, $([\alpha_1\phi(E_1) + \cdots + \alpha_n\phi(E_n) + \phi(B)]x, x)$, and thus $\alpha_1\phi(E_1) + \cdots + \alpha_n\phi(E_n) + \phi(B) = 0$. Since $(\phi(\cdot)x, x)$ is a *positive* linear functional on $C(X)$, the integral $(\phi(\alpha_1 E_1 + \cdots + \alpha_n E_n + B)x, x)$ of $\alpha_1 E_1 + \cdots + \alpha_n E_n + B$ is non-negative if $\alpha_1 E_1 + \cdots + \alpha_n E_n + B \geqq 0$, so that ϕ is order preserving. This together with the fact that $\phi(I) \leqq I$ implies that the extended ϕ has norm less than or equal to 1, and the proof is complete.

With the aid of The Generalized Schwarz Inequality we prove:

THEOREM 2. *Each isometry ρ of the Jordan algebra of self-adjoint elements in a C^*-algebra $\mathfrak{A}$ onto the Jordan algebra of self-adjoint elements in the C^*-algebra $\mathfrak{A}_1$, when extended linearly to all of $\mathfrak{A}$, has the form $\rho = U \cdot \phi$ where ϕ is a C^*-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$ and U is a self-adjoint unitary operator in the center of $\mathfrak{A}_1$, viz., $\rho(I)$.*

PROOF. Since I is an extreme point of the convex set of self-adjoint elements of norm not exceeding 1 in $\mathfrak{A}$, $\rho(I)$ is an extreme point of the corresponding set in $\mathfrak{A}_1$, (cf. [2]). An examination of the function representation of the uniformly closed algebra generated by $\rho(I)$ and I shows that $\rho(I)$ is a (self-adjoint) unitary operator U. Now the map $U \circ \rho$ takes I into I and is isometric on self-adjoint elements. The hypotheses of [Lemma 8; 2] can be weakened to include maps which are isometric on self-adjoint elements alone, and one can still conclude that if such a map preserves the identity it preserves adjoints. Indeed, all that was needed in the proof of this lemma was the fact that the map in question is isometric on operators of the form $A + inI$, A self-adjoint, n an integer (which is immediate from the hypotheses). Thus $U \circ \rho$ preserves adjoints. Since ρ maps onto the set of self-adjoint elements in $\mathfrak{A}_1$, multiplication by U sends each self-adjoint element in $\mathfrak{A}_1$ into a self-adjoint element. Thus U is in the center of $\mathfrak{A}_1$. Let ϕ be $U \circ \rho$.

It was remarked in [2] that for self-adjoint A of norm less than or equal to 1, $\| I - A \| \leqq 1$ is a necessary and sufficient condition for A to be positive. Thus ϕ preserves order (as does ϕ^{-1}). Since ϕ is onto, there exists an operator B in $\mathfrak{A}$ such that $\phi(B) = \phi(A)^2 \leqq \phi(A^2)$, so that $B \leqq A^2$. However, for ϕ^{-1}, we can assert $\phi^{-1}([\phi(A)]^2) = B \geqq [\phi^{-1}(\phi(A))]^2 = A^2$. Thus $B = A^2$, so that $\phi(A^2) = \phi(A)^2$, and the proof is complete.

It is now a simple matter to establish the following results:

COROLLARY 3. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two C^*-algebras with pure state spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively and function representations $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively on these state spaces, and if $\mathcal{P}_1$ and $\mathcal{P}_2$ are homeomorphic under a map which carries $\mathfrak{K}_1$ onto $\mathfrak{K}_2$, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-isomorphic under a map which induces the given homeomorphism.*

PROOF. In [3] the function representation mentioned in the statement of this corollary is discussed in detail. The spaces $\mathcal{P}_1$, $\mathcal{P}_2$ are the w^* closures of sets of extreme points of the positive linear functionals that are 1 at I on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. An operator in the algebra is mapped into that function whose value at a given pure state is the value of the given pure state on the operator. This representation is a linear isomorphism between the operator algebra and a linear subspace of the set of complex, continuous functions on the pure state space, which sends scalars into constants, maps the * operation into complex conjugation, and preserves the norm of self-adjoint operators. Since the given homeomorphism maps $\mathfrak{K}_1$ onto $\mathfrak{K}_2$, it sends the real functions in $\mathfrak{K}_1$ onto the real functions in $\mathfrak{K}_2$ in a linear, norm preserving manner. This induces a linear isomorphism ϕ of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ which preserves self-adjoints and their norms, and sends I onto I. Thus, by Theorem 2, ϕ is a C^*-isomorphism and induces the given homeomorphism by its very definition.

COROLLARY 4. *If $\mathfrak{A}$ is a C^*-algebra, $\mathcal{P}$ its pure state space, $\mathfrak{L}$ the function representation of $\mathfrak{A}$ on $\mathcal{P}$, and if $\mathfrak{L}$ is closed under pointwise multiplication of functions then $\mathfrak{A}$ is commutative, the representation of $\mathfrak{A}$ as $\mathfrak{L}$ is an algebraic isomorphism, and $\mathfrak{L}$ is the set of all continuous, complex-valued functions on the compact-Hausdorff space of pure states of $\mathfrak{A}$.*

PROOF. That $\mathfrak{L}$ is the full set of functions on the pure state space follows from the Stone-Weierstrass Theorem [7]. Under the given hypothesis $\mathfrak{L}$ is a commutative C^*-algebra (under pointwise multiplication) and the representation of $\mathfrak{A}$ as $\mathfrak{L}$ provides a linear isomorphism between $\mathfrak{A}$ and this C^*-algebra which is norm preserving on self-adjoint elements and takes I onto I. From Theorem 2, it follows that the representation is a C^*-isomorphism. But $\mathfrak{L}$ is commutative, and a C^*-isomorphism of a commutative algebra is an isomorphism. Thus $\mathfrak{A}$ is commutative and the representation is an isomorphism.

Of course these last results would hold if, instead of the pure state space, we used the space of all states. Our choice of the pure state space is motivated by the following considerations. In the case of a C^*-algebra generated by a single self-adjoint operator A and the identity operator I, the pure state space is completely identifiable with the spectrum of A (in fact, the algebra is isomorphic to the set of continuous functions on the spectrum of A). This space is then "the spectrum of the algebra". Two such algebras are isomorphic if and only if they have homeomorphic spectra. It is natural then to call the pure state space "the spectrum of the algebra" for arbitrary C^*-algebras. With this terminology, Corollary 3 becomes: two C^*-algebras are C^*-isomorphic if and only if they have homeomorphic spectra via a homeomorphism which preserves the representing function system. The representing function system, in its relation to the set of all continuous functions on the pure state space, measures the commutativity of the algebra (when these families of functions coincide, Corollary 4 tells us that the C^*-algebra is commutative). In giving the complete set of algebraic invariants for commutative C^*-algebras only the spectrum need be mentioned, for the equality of "the index of commutativity" is contained in the assumptions. To point this situation out most strikingly, it helps to observe that one can find two C^*-algebras (one of them commutative) with identical spectrum but which are not C^*-isomorphic; viz., any non-commutative C^*-algebra and the set of all continuous functions on its spectrum.

It is worth pointing out that Sherman's Theorem [6] is a natural and easy consequence of The Generalized Schwarz Inequality. Sherman's Theorem states that if the self-adjoint elements of a C^*-algebra form a lattice in their natural order then the algebra is commutative. In fact, if the self-adjoint elements form a lattice then they are linear lattice isomorphic to a $C(X)$, by [4]. However, in proving Theorem 2 with the aid of The Generalized Schwarz Inequality, we proved:

COROLLARY 5. *A linear order isomorphism between two C^*-algebras which carries the identity of one algebra into the identity of the other is a C^*-isomorphism.*

A linear order isomorphism is an order preserving linear isomorphism whose inverse is also order preserving. It follows then that the self-adjoint elements in the C^*-algebra of Sherman's Theorem are C^*-isomorphic with the $C(X)$. But, since the $C(X)$ is commutative, the C^*-isomorphism is an algebraic isomorphism, and the C^*-algebra is commutative. These remarks concerning Sherman's Theorem are intended only to illustrate the power of The Generalized Schwarz Inequality in dealing with order questions in operator algebras. Actually, the proof given by Sherman avoids the weighty considerations involved in The Generalized Schwarz Inequality by taking early advantage of the commutativity available at one end of the line.

In [2] and in the present paper, our key results have concerned linear isomorphisms, with additional properties, of one C*-algebra *onto* another. It is natural enough to inquire whether or not the *onto* restriction can be dropped. The general answer is very definitely no, and the function representation of a non-commutative C^*-algebra on its pure state space illustrates this. In Corollary 5, one cannot even drop the assumption that the inverse map be order-preserving as we shall illustrate by example in §3. In spite of these facts, if we assume that our linear map preserves absolute values of self-adjoint operators and sends I into I, we can drop all other restrictions on the map, even that it be an isomorphism, and conclude that it is a C^*-homomorphism. Although the assumption on absolute values appears, at first glance, to be only slightly stronger than the order preserving assumption, it should be observed that this hypothesis actually implies information concerning the set of self-adjoint operators in the algebra which commute with a given self-adjoint operator. Indeed, $A \vee 0 = \frac{1}{2}(|A| + A)$, and $A \vee 0$ is the smallest operator in the algebra which commutes with A and is greater than both A and 0. We prove:

THEOREM 6. *If ϕ is a linear map, which sends I into I, of one C^*-algebra into another, and $\phi(|A|) = |\phi(A)|$ for each self-adjoint A in the algebra then ϕ is a C^*-homomorphism.*

PROOF. If A and B are operators in our algebra

$$0 \leqq \phi[(\bar{\alpha}A^* + B^*)(\alpha A + B)] = |\alpha|^2\phi(A^*A) + 2\mathrm{Re}\{\alpha\phi(B^*A)\} + \phi(B^*B)$$

where Re $C = (C + C^*)/2$ for any operator C. Thus, for each x in the Hilbert space,

$$|\alpha|^2(\phi(A^*A)x, x) + 2\mathrm{Re}\,(\alpha\phi(B^*A)x, x) + (\phi(B^*B)x, x) \geqq 0,$$

and $2\mathrm{Re}\,(\alpha\phi(B^*A)x, x) = 2|\alpha|\,\mathrm{Re}\,(\theta\phi(B^*A)x, x)$ where $\alpha = |\alpha|\theta$. Choosing α so that $\theta = 1$, we conclude

$$(\phi(A^*A)x, x)(\phi(B^*B)x, x) \geqq |\mathrm{Re}\,(\phi(B^*A)x, x)|^2.$$

Now, if $(\phi(B^*A)x, x) = |(\phi(B^*A)x, x)|\theta_1$, we replace A by $\theta_1 A$, so that the left side of this inequality remains the same and the right becomes $|(\phi(B^*A)x, x)|^2$. Thus we have

$$(\phi(A^*A)x, x)(\phi(B^*B)x, x) \geqq |(\phi(B^*A)x, x)|^2 \tag{4}$$

for all A and B in our algebra and all x in the Hilbert space.

Suppose that $C = AB$ is a positive operator with A a positive operator. Then, by (4)

$$(\phi(A)x, x)(\phi(B^*AB)x, x) \geqq | (\phi(A^{\frac{1}{2}}A^{\frac{1}{2}}B)x, x) |^2 = | (\phi(C)x, x) |^2.$$

Since $\phi(C)$ is positive, if $\phi(A)x = 0$ then $\phi(C)x = 0$, i.e., the null space of $\phi(C)$ contains the null space of $\phi(A)$. With A a positive, regular operator in our algebra,

$$(A^2 - \lambda I) \vee 0 = [(A - \lambda^{\frac{1}{2}}I) \vee 0][A + \lambda^{\frac{1}{2}}I]$$

and

$$(A - \lambda^{\frac{1}{2}}I) \vee 0 = [(A^2 - \lambda I) \vee 0][A + \lambda^{\frac{1}{2}}I]^{-1}$$

for $\lambda \geqq 0$, so that $\phi[(A^2 - \lambda I) \vee 0]$ and $\phi[(A - \lambda^{\frac{1}{2}}I) \vee 0]$ have the same null space. However, since ϕ preserves absolute values,

$$\phi[(A^2 - \lambda I) \vee 0] = (\phi(A^2) - \lambda I) \vee 0, \qquad \phi[(A - \lambda^{\frac{1}{2}}I) \vee 0] = (\phi(A) - \lambda^{\frac{1}{2}}I) \vee 0,$$

and the null spaces of these last operators are E_λ, $F_{\lambda^{\frac{1}{2}}}$ respectively, where $\{E_\alpha\}$, $\{F_\alpha\}$ are the spectral resolutions for $\phi(A^2)$ and $\phi(A)$ respectively. Thus

$$\phi(A)^2 = \int \lambda^2 dF_\lambda = \int \lambda^2 dE_{\lambda^2} = \int \alpha dE_\alpha = \phi(A^2).$$

It follows easily, now, as in [Theorem 7; 2], that $\phi(B)^2 = \phi(B^2)$ for all operators B in the algebra, and thus ϕ is a C^*-homomorphism.

3. Remarks and examples

The crucial tool for our investigation was The Generalized Schwarz Inequality. One might wonder whether the standard proof of the ordinary Schwarz Inequality would apply to our general situation to give a simple proof. Unfortunately this is not the case. Indeed, the familiar argument yields $0 \leqq \phi[(\alpha A + I)^2] = \alpha^2\phi(A^2) + 2\alpha\phi(A) + I$ for A self-adjoint and α real, from which we would wish to conclude that $\phi(A^2) \geqq \phi(A)^2$ as in the case where $\phi(A^2)$ and $\phi(A)$ are real numbers. When $\phi(A^2)$ and $\phi(A)$ commute, one can reduce the problem to the real-valued case, and the result follows from these considerations. When $\phi(A^2)$ and $\phi(A)$ do not commute, one cannot conclude the desired inequality by these means as the following example shows. Choose $B = \begin{pmatrix} 2 & 2^{\frac{1}{2}} \\ 2^{\frac{1}{2}} & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; an easy computation then shows that $\alpha^2 B + 2\alpha C + I \geqq 0$ for all real α, however, $B \ngeqq C^2$. The non-commutativity of the image of ϕ forces us to abandon the usual techniques.

Our principal reason for developing The Generalized Schwarz Inequality was for its application to Theorem 2 and thence to Corollary 3. It should be noted that a proof of these last results, in the case of rings of operators (or, more generally, whenever the C^*-algebra contains an abundance of projections), is provided in [Theorem 7; 2] by the alternative ending for the proof of Theorem 7 given in the ring of operators case. Can one then deduce these results for the arbitrary C^*-algebra from the case where there are an abundance of projections by some device such as that employed in the proof of The Generalized Schwarz Inequality to bring idempotents into the picture? One of the means for assuring the existence of sufficiently many projections in a C^*-algebra is to require that the projection on the closure of the range of an operator be in the algebra (cf. [5]). We shall produce an example of two (abelian) C^*-algebras, a $*$-isomorphism between them, and a positive operator in one of them such that the isomorphism cannot be extended to the algebra generated by the projection on the closure of the range of this operator and the original algebra. It follows that the "extension" procedure cannot be used to prove Theorem 2 and Corollary 3. For our example, let $E_0, E_1, E_2, \cdots$ be an infinite set of orthogonal non-zero projections with sum I, and let $\mathfrak{A}_1$ be the set of operators of the form $\alpha_0 E_0 + \sum_{i=1}^{\infty} \alpha_i E_i$ where (α_i) is a convergent sequence of complex numbers and α_0 is its limit. For $\mathfrak{A}_2$ we take the algebra $\sum_{i=0}^{\infty} \alpha_i E_i$ with (α_i) a convergent sequence of complex numbers ($\mathfrak{A}_2$ contains $\mathfrak{A}_1$). Both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-algebras. Define a map ϕ of $\mathfrak{A}_2$ onto $\mathfrak{A}_1$ as follows:

$$\phi\left(\sum_{i=0}^{\infty} \alpha_i E_i\right) = \alpha E_0 + \sum_{i=1}^{\infty} \alpha_{i-1} E_i$$

where α is the limit of the sequence (α_i). The map ϕ is clearly a $*$-isomorphism of $\mathfrak{A}_2$ onto $\mathfrak{A}_1$. If (α_i) is a sequence of positive reals which converges to 0 then $A = \sum_{i=0}^{\infty} \alpha_i E_i$ is a positive operator in $\mathfrak{A}_2$ with (0) nullspace. However, $\phi(A) = \sum_{i=1}^{\infty} \alpha_{i-1} E_i$ has E_0 as the projection on its null space and, hence, $I - E_0$ as the projection on the closure of its range. If ϕ^{-1} were extendable from $\mathfrak{A}_1$ to

$$\{\mathfrak{A}_1, I - E_0\}$$

then one would have

$$A = \phi^{-1}(\phi(A)) = \phi^{-1}((I - E_0)\phi(A)) = (I - \phi^{-1}(E_0))A = A - \phi^{-1}(E_0)A,$$

so that $\phi^{-1}(E_0)$ is contained in the projection on the null space of A and is therefore 0.

It should be noted that, for the purposes of Corollary 3, Theorem 2 was actually necessary and [Theorem 7; 2] was not applicable. To illustrate this, we shall exhibit a C^*-algebra and an operator therein whose representing function on the pure state space of the algebra does not have the same norm as the operator (of course the operator will have to be non-normal). In fact, let the C^*-algebra be all 2 x 2 complex matrices. The pure states of this algebra are all given by vectors x of the underlying 2-dimensional unitary space, and the value at a given operator A is (Ax, x). The operator $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has norm 1 but the norm of its representing function on the pure state space is $\sup \{\beta(1 - \beta^2)^{\frac{1}{2}} : 0 \leqq \beta \leqq 1\} = \frac{1}{2}$.

Our final example will show that the hypotheses of Corollary 5 cannot be relaxed to allow an order preserving linear isomorphism in place of the linear order isomorphism, i.e., we shall give an example of an order preserving linear isomorphism of one C^*-algebra onto another whose inverse is not order preserving (or equivalently, by Corollary 5, which is not a C^*-isomorphism). For both our C^*-algebras, we choose the set of complex functions on a two point space (that is, pairs of complex numbers under coordinatewise multiplication and addition). Our linear isomorphism is given by the matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ acting on the complex number pairs in the standard fashion. This linear transformation is an isomorphism, since its determinant is non-zero; it clearly takes positive functions into positive functions, the identity function into the identity function, and real functions into real functions; however, it takes the non-positive function $(1, -1)$ into the positive function $(0, \frac{1}{2})$.

One can check directly that the proof of The Generalized Schwarz Inequality remains valid (with trivial modifications) when ϕ is an order preserving linear map with norm less than or equal to 1 of a C^*-algebra which doesn't contain the identity operator I. However, it is a simple matter to prove the inequality in this case by extending ϕ to the algebra $\mathfrak{A}_1$ generated by our algebra $\mathfrak{A}$ and I. In fact, define: $\phi(\alpha I + A) = \alpha I + \phi(A)$ for A in the algebra. If $\alpha I + A \geqq 0$ then $\alpha \geqq 0$, for otherwise A would have an inverse and $\mathfrak{A}$ contain I contrary to assumption. Thus $\alpha I \geqq 0$,and $\alpha I \geqq -A$ so that $\alpha I \geqq (-A) \vee 0$ from which $\alpha \geqq \| (-A) \vee 0 \|$. Since ϕ has norm less than or equal to 1, $\alpha \geqq \| \phi\, [(-A) \vee 0] \|$ or $\alpha I \geqq \phi[(-A) \vee 0]$. Now $\phi[(-A) \wedge 0]$ is a negative operator, since ϕ is order preserving, and thus $\alpha I \geqq \phi[(-A) \vee 0] + \Phi\, [(-A) \wedge 0] = \phi(-A) = -\phi(A)$. Thus ϕ, as extended, is order preserving. With this extension of The Generalized Schwarz Inequality, Corollary 5 goes through in the non-unit situation with the added assumption that the map φ as well as ϕ^{-1} has norm less than or equal to 1, i.e., the assumption that ϕ is isometric. In this form Corollary 5 is probably the natural generalization of Theorem 2 to the non-unit situation. In the non-unit case there is no canonical procedure, such as examining the image of the identity operator, for telling whether or not the given isometry is in normalized form. When an identity is present, the order preserving assumption is easily seen to be equivalent to the assumption that the identity goes into the identity. Undoubtedly, more can be said about the non-normalized isometries in the non-unit case. At any rate, Corollary 5, as extended to the non-unit case, suffices to establish Corollary 3 in the non-unit case.

In conclusion, we may remark that, even in the case where the self-adjoint algebras studied are not uniformly closed, the various assumptions on the maps considered assure their uniform continuity and hence their extendability to the uniform closure of the algebras. By these means, our results can be applied to not necessarily closed self-adjoint operator algebras.

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